An extension of Mizoguchi–Takahaashi's fixed point theorem

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Abstract. Our main theorem is an extension of the well–known Mizoguchi–Takahaashi's fixed point theorem [N. Mizogochi and W. Takahashi, Fixed point theorems for multi–valued mappings on complete metric space, *J. Math. Anal. Appl.* 141 (1989) 177–188].

1. INTRODUCTION AND STATEMENT OF RESULTS

Let (X, d) be a metric space. $CB(X)$ denotes the collection of all nonempty closed bounded subsets of X. For $A, B \in CB(X)$, and $x \in X$, define $D(x, A) := \inf\{d(x, a); a \in A\}$, and

$$H(A, B) := \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A).$$

It is easy to see that H is a metric on $CB(X)$. H is called the Hausdorff metric induced by d.

Definition 1.1. *An element $x \in X$ is said to be a fixed point of a multi–valued mapping $T : X \to CB(X)$, if such that $x \in T(x)$.*

One can show that $(CB(X), H)$ is a complete metric space, whenever (X, d) is a complete metric space (see for example Lemma 8.1.4, of [8]).

In 1969, Nadler [5] extended the Banach contraction principle [1] to set–valued mappings as follows.

Theorem 1.2. *Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0, 1)$ such that $\mathcal{H}_d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in T(z)$.*

Nadler's theorem was generalized by Mizoguchi and Takahaashi [4] in the following way.

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Theorem 1.3. *Let (X,d) be a complete metric space and let T be a mapping from (X,d) into $(CB(X),H)$ satisfies*

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$$

for all $x,y \in X$, where α be a function from $[0,\infty)$ into $[0,1)$ such that $\limsup_{s\to t^+}\alpha(s) < 1$ for all $t \in [0,\infty)$. Then T has a fixed point.

Recently Suzuki [9] proved the Mizoguchi–Takahashi's fixed point theorem by an interesting and short proof.

On the other hand, Banach contraction principle was generalized by Reich [6, 7] as follows.

Theorem 1.4. *Let (X,d) be a complete metric space and let T be a mapping from (X,d) into $(CB(X),H)$ satisfies*

$$H(Tx,Ty) \leq \beta[D(x,Tx) + D(y,Ty)]$$

for all $x,y \in X$, where $\beta \in [0,\frac{1}{2})$. Then T has a fixed point.

In 1973, Hardy and Rogers [3] extended the Reich's theorem by the following way.

Theorem 1.5. *Let (X,d) be a complete metric space and let T be a mapping from X into X such that*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma[d(x,Ty) + d(y,Tx)]$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

Recently, the authors of the present paper [2] extended the theorems 1.5 and 1.2 as follows.

Theorem 1.6. *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that*

$$H(Tx,Ty) \leq \alpha d(x,y) + \beta[D(x,Tx) + D(y,Ty)] + \gamma[D(x,Ty) + D(y,Tx)]$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

In this paper, we shall generalize above results. More precisely, we prove the following theorem, which can be regarded as an extension of all theorems 1.2,1.3,1.4,1.5 and 1.6.

Theorem 1.7. *Let (X, d) be a complete metric space and let T be mapping from X into $CB(X)$ such that*

$$\begin{aligned} H(Tx, Ty) &\leq \alpha(d(x, y))d(x, y) + \beta(d(x, y))[D(x, Tx) + D(y, Ty)] \\ &\quad +\gamma(d(x, y))[D(x, Ty) + D(y, Tx)] \end{aligned}$$

for all $x, y \in X$, where α, β, γ are mappings from $[0, \infty)$ into $[0, 1)$ such that $\alpha(t)+2\beta(t)+2\gamma(t) < 1$ and $\limsup_{s\to t^+} \frac{\alpha(t)+\beta(t)+\gamma(t)}{1-(\beta(t)+\gamma(t))} < 1$ for all $t \in [0, \infty)$. Then T has a fixed point.

Moreover, we conclude the following results by using theorem 1.7.

Corollary 1.8. *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfies*

$$H(Tx, Ty) \leq \beta(d(x, y))[D(x, Tx) + D(y, Ty)]$$

for all $x, y \in X$, where β be a function from $[0, \infty)$ into $[0, \frac{1}{2})$ and $\limsup_{s\to t} \beta(s) < \frac{1}{2}$ for all $t \in [0, \infty)$. Then T has a fixed point.

Corollary 1.9. *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfies*

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + \beta(d(x, y))[D(x, Tx) + D(y, Ty)]$$

for all $x, y \in X$, where α, β are function from $[0, \infty)$ into $[0, 1)$ such that $\alpha(t)+2\beta(t) < 1$ and $\limsup_{s\to t^+} (\frac{\alpha(t)+\beta(t)}{1-\beta(t)}) < 1$ for all $t \in [0, \infty)$. Then T has a fixed point.

2. Proof of the main theorem

Proof. Define function α' from $[0, \infty)$ into $[0, 1)$ by $\alpha'(t) = \frac{\alpha(t)+1-2\beta(t)-2\gamma(t)}{2}$ for $t \in [0, \infty)$. Then we have the following assertions:

1) $\alpha(t) < \alpha'(t)$ for all $t \in [0, \infty)$.

2) $\limsup_{s\to t^+} \frac{\alpha'(t)+\beta(t)+\gamma(t)}{1-(\beta(t)+\gamma(t))} < 1$ for all $t \in [0, \infty)$.

3) For $x, y \in X$ and $u \in Tx$, there exists $\nu \in Ty$ such that

$$\begin{aligned} d(\nu, u) &\leq \alpha'(d(x, y))d(x, y) + \beta(d(x, y))[D(x, Tx) + D(y, Ty)] \\ &\quad +\gamma(d(x, y))[D(x, Ty) + D(y, Tx)]. \end{aligned}$$

Putting $u = y$ in 3), we obtain that:
4) For $x \in X$ and $y \in Tx$ there exists $\nu \in Ty$ such that

$$\begin{aligned} d(\nu, y) &\leq \alpha^{'}(d(x,y))d(x,y) + \beta(d(x,y))[D(x,Tx) + D(y,Ty)] \\ &\quad + \gamma(d(x,y))[D(x,Ty) + D(y,Tx)]. \end{aligned}$$

Hence, we can define sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{n+1} \in Tx_n, x_{n+1} \neq x_n$ and

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\leq \alpha^{'}(d(x_{n+1},x_n))d(x_{n+1},x_n) + \beta(d(x_{n+1},x_n))[D(x_n,Tx_n) \\ &\quad + D(x_{n+1},Tx_{n+1})] + \gamma(d(x_{n+1},x_n)[D(x_n,Tx_{n+1}) \\ &\quad + D(x_{n+1},Tx_n)] \end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$d(x_{n+2}, x_{n+1}) \leq \frac{\alpha^{'}(d(x_{n+1},x_n)) + \beta(d(x_{n+1},x_n)) + \gamma(d(x_{n+1},x_n))}{1 - (\beta(d(x_{n+1},x_n)) + \gamma(d(x_{n+1},x_n)))} d(x_{n+1},x_n)$$

for all $n \in \mathbb{N}$. On the other hand, we have

$$\frac{\alpha^{'}(t) + \beta(t) + \gamma(t)}{1 - (\beta(t) + \gamma(t))} < 1$$

for all $t \in\ [0,\infty)$, then $\{d(x_{n+1},x_n)\}$ is a non-increasing sequence in $\mathbb{R}$. Hence, $\{d(x_{n+1},x_n)\}$ is a converges to some nonnegative integer τ. By assumption,

$$\limsup_{s\to\tau^+} \frac{\alpha^{'}(s) + \beta(s) + \gamma(s)}{1 - (\beta(s) + \gamma(s))} < 1$$

so, we have

$$\frac{\alpha^{'}(\tau) + \beta(\tau) + \gamma(\tau)}{1 - (\beta(\tau) + \gamma(\tau))} < 1$$

then, there exist $r \in [0,1)$ and $\epsilon > 0$ such that

$$\frac{\alpha^{'}(s) + \beta(s) + \gamma(s)}{1 - \beta(s) + \gamma(s)} < r$$

for all $s \in [\tau, \tau + \epsilon]$. We can take $\nu \in \mathbb{N}$ such that

$$\tau \leq d(x_{n+1},x_n) \leq \tau + \epsilon$$

for all $n \in \mathbb{N}$ with $n \geq \nu$. It follows that

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &\leq \frac{\alpha^{'}(d(x_{n+1},x_n)) + \beta(d(x_{n+1},x_n)) + \gamma(d(x_{n+1},x_n))}{1 - (\beta(d(x_{n+1},x_n)) + \gamma(d(x_{n+1},x_n)))} d(x_{n+1},x_n) \\ &\leq r d(x_{n+1},x_n) \end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq \nu$. This implies that

$$\sum_{n=1}^{\infty} d(x_{n+2}, x_{n+1}) \leq \sum_{n=1}^{\nu} d(x_{n+1}, x_n) + \sum_{n=1}^{\infty} r^n d(x_{\nu+1}, x_\nu) < \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, then $\{x_n\}$ converges to some point $x^* \in X$. Now, we have

$$\begin{aligned}
D(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + D(x_{n+1}, Tx^*) \\
&\leq d(x^*, x_{n+1}) + H(Tx_n, Tx^*) \\
&\leq d(x^*, x_{n+1}) + \alpha(d(x_n, x^*))d(x_n, x^*) \\
&+ \beta(d(x_n, x^*))[D(x_n, Tx_n) + D(x^*, Tx^*)] \\
&+ \gamma(d(x_n, x^*))[D(x_n, Tx^*) + D(x^*, Tx_n)]
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
D(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + \alpha(d(x_n, x^*))d(x_n, x^*) \\
&+ \beta(d(x_n, x^*))[d(x_{n+1}, x_n) + D(x^*, Tx^*)] \\
&+ \gamma(d(x_n, x^*))[D(x_n, Tx^*) + d(x_n, x^*)]
\end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}
D(x^*, Tx^*) &\leq \liminf_{n\to\infty}(\beta(d(x_n, x^*)) + \gamma(d(x_n, x^*)))D(x^*, Tx^*) \\
&= \liminf_{s\to 0^+}(\beta(s) + \gamma(s))D(x^*, Tx^*) \\
&\leq \limsup_{s\to 0^+}(\frac{\alpha(s) + \beta(s) + \gamma(s)}{1 - (\beta(s) + \gamma(s))})D(x^*, Tx^*).
\end{aligned}$$

On the other hand, we have

$$\limsup_{s\to 0^+}(\frac{\alpha(s) + \beta(s) + \gamma(s)}{1 - (\beta(s) + \gamma(s))}) < 1$$

then $D(x^*, Tx^*) = 0$. Since Tx^* is closed, then $x^* \in Tx^*$. □

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